Course 16
Geometric Data Structures for Computer Graphics

Voronoi Diagrams

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Definition Voronoi Diagram

Classical Voronoi Diagram in 2-D:

- Set of sites $S$ in the Euclidean plane
- Subdivision into regions of the same neighborship
- Well-known concept in Biology, Economics, CS, ...
Abstract definition

- **Bisector:** $B(p, q) = \{x \mid d(p, x) = d(q, x)\}$
- **Halfplane:** $D(p, q) = \{x \mid d(p, x) < d(q, x)\}$
- **Voronoi Region:** $\text{VR}(p, S) = \bigcap_{q \in S, q \neq p} D(p, q)$
- **Voronoi Diagram:** $V(S) = \bigcup_{p, q \in S, p \neq q} \text{VR}(p, S) \cap \text{VR}(q, S)$
Properties

- Voronoi Diagram
  - Graph
  - Complexity: $O(n)$ edges and vertices,
  - Region: 6 boundary edges in the average (Application of Euler-Formula)
  - Data Structure: DCEL, Adjacency List
  - Simple linear structure, represents a decomposition of the plane in cells
  - Implementations: LEDA, CGAL, Qhull, ...
Simple Applications

Voronoi Diagram of a set of points is given

1. All Nearest Neighbors
   - \(O(n)\) time

2. Closest Pair
   - \(O(n)\) time

3. Post Office Problem/Locus Approach
   - Query time: \(O(\log n)\)
   - Simple preprocessing: \(O(n^2)\) time and space
   - More complex: \(O(n)\) (Edelsbrunner)
Delaunay Triangulation: The Dual

- The dual graph $D_T(S)$
- Triangulation of $S$, $(n - 1)$ triangles
- Characterizations
  - Triangle: Circumcircle contains no other site
  - Edge: Circle contains no other site
- Maximizes the minimum angle
Computation

- Lower bound: $\Omega(n \log n)$
  - Reduction to the Convex Hull (Shamos)
  - Reduction to $\epsilon$-closeness (Zhu and Mirzaian)

- Construction: $O(n \log n)$
  - Incremental
  - Divide and Conquer
  - Sweep
  - Delaunay Triangulation
Simple Incremental Construction

- Works on the Delaunay Triangulation
- Easy to implement/generalize
- Using edge flips
- Assume that $DT(\{p_1, p_2, \ldots, p_{i-1}\})$ was constructed
- Insert $p_i$
- Conflicts with Delaunay triangles

\[ q \]
\[ p \]
\[ r \]
\[ s \]
Simple Incremental Construction

Insert $p_i$

- Determine triangle
- Successively remove conflicts by Edge-flips
More Applications

Assume that the diagram is given:

- $k$-th nearest neighbor of point $x \notin S$: $O(k \log^2 n)$ expected time
- Minimum Spanning Tree, TSP-Heuristic: $O(n \log n)$
- Largest empty circle in area $A$: $O(n)$
- Smallest enclosing circle/square: $O(n)$
- Localization problems (Hamacher)
- Clustering of objects (Dehne, Noltemeier)
Voronoi Diagram in 3-D

Set of points in the Euclidean 3D Space

- Bisector: Hyperplane
- Region: Intersection of halfspaces bounded by bisectors, 3D convex polyhedron
- Boundary of region: Facets, edges, vertices
- Decomposition of the space into 3D convex cells
Voronoi Diagram in 3-D

- Delaunay triangulation
  - Tetrahedron for every vertex
  - Triangle for every edge
  - Edge for every facet
  - Delaunay Tetrahedron: Circumsphere of four points is empty
- Unfortunately no demo software :-((
Voronoi Diagram in 3-D

- **Complexity:**
  - $O(n^2)$
  - Uniformly distributed: $O(n)$

- **Construction:**
  - Similar incremental approach: 3D edge flips in $O(n^2)$

Two–into–three tetrahedra flip for five sites
Voronoi Diagram in 3-D

Application:

- Generalizations of 2-D applications
- For example: Post Office Problem, Smallest enclosing ball, All nearest neighbors, etc.
Other generalizations

Other metrics

- $L_1$-Metric ($L_\infty$-Metric)
- Convex distance functions

More generalizations: weights, other objective (z.B. farthest points), colors, ...